Symplectic Covariance Properties for Shubin and Born–Jordan Pseudo-Differential Operators

Maurice A. de Gosson
University of Vienna
Faculty of Mathematics - NuHAG
Nordbergstr. 15, 1090 Vienna

January 20, 2013

Abstract

Among all classes of pseudo-differential operators only the Weyl operators enjoy the property of symplectic covariance with respect to conjugation by elements of the metaplectic group. In this paper we show that there is, however, a weaker form of symplectic covariance for Shubin's τ-dependent operators, in which the intertwiners no longer are metaplectic, but still are invertible non-unitary operators. We also study the case of Born–Jordan operators, which are obtained by averaging the τ-operators over the interval $[0,1]$ (such operators have recently been studied by Boggiatto and his collaborators). We show that metaplectic covariance still hold for these operators, with respect top a subgroup of the metaplectic group.

1 Introduction

In the early years of quantum mechanics physicists were confronted with an ordering problem: assume that some quantization process associated to the real variables x (position) and p (momentum) two operators $\widehat{X}$ and $\widehat{P}$ satisfying the canonical commutation rule $\widehat{X_j}\widehat{P_j} - \widehat{P_j}\widehat{X_j} = i\hbar$. What should then be the operator associated to the monomial $x^m p^n$? The first to give a mathematically motivated answer was Weyl [20]; he was was developing his ideas on a group theoretical approach to quantization which lead to the prescription

$$x_j^m p_j^\ell \overset{\text{Weyl}}{\longrightarrow} \frac{1}{2^\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} \widehat{P_j}^{\ell-k} \widehat{X_j}^m \widehat{P_j}^k \tag{1}$$

It turns out that the Weyl ordering is a particular case of the more general "τ-ordering": for any real number τ one defines

$$x_j^m p_j^\ell \overset{\tau}{\longrightarrow} \sum_{k=0}^{\ell} \binom{\ell}{k} (1-\tau)^k \tau^{\ell-k} \widehat{P_j}^k \widehat{X_j}^\ell \widehat{P_j}^{\ell-k} \tag{2}$$

which reduces to Weyl's prescription when $\tau = \frac{1}{2}$. We will from now assume that $\widehat{X_j} f = x_j f$ and $\widehat{P_j} f = -2\pi i \partial_{x_j} f$. The τ-ordering (2) is itself a particular case of the Shubin pseudo-differential calculus (Shubin [17]): given a symbol a the τ-pseudo-differential operator $A_\tau = \mathrm{Op}_\tau(a)$ is formally defined by

$$A_\tau f(x) = \iint e^{2\pi i p(x-y)} a(\tau x + (1-\tau)y, p) f(y) dp dy;$$

for $\tau = \frac{1}{2}$ we recover the Weyl correspondence. Using Schwartz's kernel theorem it is not difficult to show that for every continuous linear operator $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ and for every $\tau \in \mathbb{R}$ there exists $a \in \mathcal{S}'(\mathbb{R}^{2n})$ such that $A = \mathrm{Op}_\tau(a)$; the τ-operators are thus of a very general nature. Now, it is (reasonably) well-known (Stein [18], Wong [21]) that among all τ-operators only the Weyl operators enjoy a symmetry property known as "symplectic covariance":

> *If* $\mathrm{Op}_\tau(a \circ S) = \widehat{S}\,\mathrm{Op}_\tau(a)\widehat{S}^{-1}$ *for every* $\widehat{S} \in \mathrm{Mp}(2n, \mathbb{R})$ *then* $\tau = \frac{1}{2}$.

Here $\mathrm{Mp}(2n, \mathbb{R})$ is the metaplectic group and S the projection of $\widehat{S} \in \mathrm{Mp}(2n, \mathbb{R})$ on the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. Symplectic covariance in the sense above is thus a *characteristic property* of Weyl pseudo-differential calculus. In fact, one shows more generally (Stein [18], §12.7, Wong, Chapter 30) that:

> *Let* $a \longmapsto \mathrm{Op}(a)$ *be a linear mapping from* $\mathcal{S}'(\mathbb{R}^{2n})$ *to the space of linear operators that is continuous in the topology of* $\mathcal{S}'(\mathbb{R}^{2n})$. *Assume that: (i) if* $a = a(x)$, $a \in L^\infty(\mathbb{R}^n)$, *then* $\mathrm{Op}(a)$ *is multiplication by* $a(x)$; *(ii) if* $S \in \mathrm{Sp}(2n, \mathbb{R})$ *then* $\mathrm{Op}(a \circ S) = \widehat{S}\,\mathrm{Op}(a)\widehat{S}^{-1}$. *Then* $a \longmapsto \mathrm{Op}(a)$ *is the Weyl correspondence*

so the property of symplectic covariance really singles out Weyl operators among all possible "quantization schemes".

The principal aim of this paper is to report on the fact that there exists a weaker form of symplectic covariance for τ-operators extending which

reduces to the case above when $\tau = \frac{1}{2}$. In fact, we will show in Proposition 4 that to each $S \in \mathrm{Sp}(2n, \mathbb{R})$ one can attach an invertible operator $R_\tau(S) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ such that

$$R_\tau(S)\, \mathrm{Op}_\tau(a) = \mathrm{Op}_\tau(a \circ S) R_\tau(S). \tag{3}$$

These operators are in general not unitary, and do thus not generate a copy of $\mathrm{Mp}(2n, \mathbb{R})$.

As a consequence of our constructions we will be able to establish a similar property for Born–Jordan pseudo-differential operators. These operators were very recently introduced in de Gosson and Luef [12] where it was remarked that the constructions of Boggiatto and his collaborators [1, 2, 3] of a certain pseudo-differential class was related to a quantization procedure going back to Born and Jordan [4] and historically anterior to the work of Weyl [20]. Born and Jordan's quantization is based on the prescription

$$x^m p^\ell \overset{\mathrm{BJ}}{\longrightarrow} \frac{1}{\ell + 1} \sum_{k=0}^{\ell} \widehat{P}^{\ell-k} \widehat{X}^m \widehat{P}^k; \tag{4}$$

an elementary calculation shows that this correspondence is obtained by averaging the τ-ordering (2) over the interval $[0, 1]$. This suggests to define more generally the Born–Jordan pseudo-differntial operator with symbol a by the formula

$$A_{\mathrm{BJ}} = \int_0^1 A_\tau d\tau.$$

We will see that the symplectic covariance formula (3) can be used to derive a similar formula for A_{BJ}.

In a recent paper de Gosson and Luef [12] have shown that this calculus corresponds to a generalization of an early quantization scheme due to Born and Jordan, and which has been largely superseded by the more elegant Weyl quantization procedure. Both Weyl and Born–Jordan quantization hark back to the early years of quantum mechanics.

Notation 1 *The Euclidean scalar product of two vectors u and v on $\mathbb{R}^m$ is denoted indifferently $u \cdot v$ or by uv. When X is a symmetric matrix we will often write Xu^2 for $Xu \cdot u$. The standard symplectic form σ on $\mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{R}^{2n}$ is defined by $\sigma(z, z') = px' - p'x$ if $z = (x, p)$, $z' = (x', p')$ the corresponding symplectic group is $\mathrm{Sp}(2n, \mathbb{R})$. We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ its dual (the tempered distributions). The normalizations we use correspond to that*

familiar from the theory of pseudo-differential operators; for instance the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is

$$Ff(x) = \int e^{-2\pi i x x'} f(x') dx'$$

(it corresponds to the choice $\hbar = 1/2\pi$ in the quantum-mechanical literature).

Acknowledgements. This work has been financed by the Austrian Research Agency FWF (Projektnummer P20442-N13).

2 The Shubin Calculus

2.1 Definitions and main properties

2.1.1 The pseudo-differential operators A_τ

The τ-pseudo-differential operator $A_\tau = \mathrm{Op}_\tau(a)$ with symbol $a \in \mathcal{S}'(\mathbb{R}^n)$ is, by definition, the operator with distributional kernel

$$K_\tau(x,y) = F_2^{-1}\left[a(\tau x + (1-\tau)y, \cdot)\right](x-y) \tag{5}$$

where F_2^{-1} is the inverse Fourier transform in the second set of variables. We can thus write formally (Shubin [17])

$$A_\tau f(x) = \iint e^{2\pi i p(x-y)} a(\tau x + (1-\tau)y, p) f(y) dp dy. \tag{6}$$

One easily verifies using this expression that the (formal) adjoint of $A_\tau = \mathrm{Op}_\tau(a)$ is given by

$$\mathrm{Op}_\tau(a)^* = \mathrm{Op}_{1-\tau}(\overline{a}). \tag{7}$$

2.1.2 The operators $\widehat{T}_\tau(z)$

Let $\widehat{T}(z_0)$ be the Heisenberg operator: it is defined for $f \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\widehat{T}(z_0) f(x) = e^{2\pi i (p_0 x - \frac{1}{2} p_0 x_0)} f(x - x_0) \tag{8}$$

where $z_0 = (x_0, p_0)$. Let τ be a real parameter and set, more generally,

$$\widehat{T}_\tau(z_0) f(x) = e^{2\pi i (p_0 x - (1-\tau) p_0 x_0)} f(x - x_0) \tag{9}$$

that is, equivalently,

$$\widehat{T}_\tau(z_0) = e^{i\pi(2\tau - 1) p_0 x_0} \widehat{T}(z_0). \tag{10}$$

We have $\widehat{T}_{1/2}(z_0) = \widehat{T}(z_0)$, and

$$\widehat{T}_\tau(z_0)^{-1} = \widehat{T}_{1-\tau}(-z_0). \tag{11}$$

It is immediate to check the following relations:

$$\widehat{T}_\tau(z_0)\widehat{T}_\tau(z_1) = e^{2\pi i\sigma(z_0,z_1)}\widehat{T}_\tau(z_1)\widehat{T}_\tau(z_0) \tag{12}$$

$$\widehat{T}_\tau(z_0+z_1) = e^{-i\pi\sigma(z_0,z_1)}\widehat{T}_\tau(z_0)\widehat{T}_\tau(z_1). \tag{13}$$

For many purposes it is useful to write formula (6) in terms of the operators $\widehat{T}_\tau(z)$:

$$A_\tau f = \mathrm{Op}_\tau(a)f = \int a_\sigma(z)\widehat{T}_\tau(z)f dz \tag{14}$$

where a_σ is the symplectic Fourier transform of a, that is

$$a_\sigma(z) = \int e^{-2\pi i\sigma(z,z')}a(z')dz'.$$

Following the usage in the theory of Weyl operators, we will call a_σ the "twisted symbol of A_τ". The distributional kernel of A_τ can then be written

$$K_\tau(x,y) = F_2^{-1}\left[a_\sigma(x-y,\cdot)\right](\tau x + (1-\tau)y) \tag{15}$$

it is often more suitable for calculations than formula (5).

2.1.3 A composition formula

The τ-operators can be composed exactly in the same way as usual Weyl operators:

Proposition 2 *Let A_τ and B_τ be given by*

$$A_\tau = \int a_\sigma(z)\widehat{T}_\tau(z)dz \quad \textit{and} \quad B_\tau = \int b_\sigma(z)\widehat{T}_\tau(z)dz. \tag{16}$$

Then, if $A_\tau B_\tau$ is defined, we have $A_\tau B_\tau = C_\tau$ with

$$c_\sigma(z) = \int e^{i\pi\sigma(z,z')}a_\sigma(z-z')b_\sigma(z')dz'. \tag{17}$$

Proof. We have

$$A_\tau B_\tau = \iint a_\sigma(z_0)b_\sigma(z_1)\widehat{T}_\tau(z_0)\widehat{T}_\tau(z_1)dz_0dz_1$$

and hence, using formula (13)

$$A_\tau B_\tau = \iint e^{i\pi\sigma(z_0,z_1)}a_\sigma(z_0)b_\sigma(z_1)\widehat{T}_\tau(z_0+z_1)dz_0dz_1.$$

The composition formula (17) follows making the change of variables $z = z_0 + z_1$, $z' = z$. $\blacksquare$

2.1.4 Relation with the τ-Wigner transform

Boggiatto and his collaborators [1, 2, 3] have recently introduced a τ-dependent Wigner transform $W_\tau(f,g)$ related with the Shubin τ-pseudo-differential calculus. Averaging over τ in the interval $[0,1]$ leads to an element of the Cohen class, i.e. to a transform of the type $Q(f,g) = W_\tau(f,g) * \theta$ where $\theta \in \mathcal{S}'(\mathbb{R}^{2n})$.

Following result relates the operator A_τ to the τ-Wigner transform:

Proposition 3 *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. We have the formula*

$$(A_\tau f|g)_{L^2} = \langle a, W_\tau(f,g)\rangle \tag{18}$$

where $W_\tau(f,g)$ is the τ-dependent cross-Wigner transform of (f,g) defined by

$$W_\tau(f,g)(z) = \int e^{-2\pi i y p} f(x+\tau y)\overline{g(x-(1-\tau)y)}dy. \tag{19}$$

Proof. We have

$$\langle a, W_\tau(\psi,\phi)\rangle = \int e^{-2\pi i y p} a(z)\psi(x+\tau y)\overline{\phi(x-(1-\tau)y)}dydpdx;$$

setting $x+\tau y = y'$, $x-(1-\tau)y = y'$ we get

$$\langle a, W_\tau(\psi,\phi)\rangle = \int e^{-2\pi i(x'-y')p} a((1-\tau)x'+\tau y', p)\psi(y')\overline{\phi(x')}dydpdx$$

hence the equality (18) in view of (6). ∎

Formula (18) yields an alternative definition of the operator $A_\tau f$ for an arbitrary symbol $a \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$: choose $g \in \mathcal{S}(\mathbb{R}^n)$; then$W_\tau(f,g) \in \mathcal{S}(\mathbb{R}^{2n})$ and the distributional bracket $\langle a, W_\tau(f,g)\rangle$ is thus defined; by definition A_τ is the the continuous operator $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ defined by the right hand-side of (18).

We notice that The τ-dependent Wigner transform $W_\tau\psi = W_\tau(\phi,\psi)$ satisfies the same marginal properties as the ordinary Wigner transform: for every $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we have

$$\int W_\tau f(x,p)dp = |f(x)|^2 \ , \ \int W_\tau\psi(x,p)dx = |Ff(p)|^2. \tag{20}$$

(see Boggiatto et al. [1]).

In the case $\tau = 1$ the transform W_τ reduces to the Rihaczek distribution, and when $\tau = 1$ we get the dual Rihaczek distribution.

3 Symplectic Covariance in Shubin Calculus

3.1 A class of intertwining operators

3.1.1 The symplectic Cayley transform

We will use the following notation:

$$\mathrm{Sp}_{(0)}(2n,\mathbb{R}) = \{S \in \mathrm{Sp}(2n,\mathbb{R}) : \det(S-I) \neq 0\}$$
$$\mathrm{Sym}_{(0)}(2n,\mathbb{R}) = \{M \in \mathrm{Sym}(2n,\mathbb{R}) : \det(M-\tfrac{1}{2}J) \neq 0\}.$$

Let $S \in \mathrm{Sp}_{(0)}(2n,\mathbb{R})$; by definition the symplectic Cayley transform (introduced in de Gosson [8, 9, 10, 11]) of S is the symmetric matrix given by

$$M(S) = \tfrac{1}{2}J(S+I)(S-I)^{-1} \tag{21}$$

(the symmetry of $M(S)$ is readily verified using the relation $S^TJS = SJS^T = J$, which is equivalent to $S \in \mathrm{Sp}(2n,\mathbb{R})$). The mapping $M(\cdot)$ is a bijection $\mathrm{Sp}_{(0)}(2n,\mathbb{R}) \longrightarrow \mathrm{Sym}_{(0)}(2n,\mathbb{R})$ and the inverse of that bijection is given by

$$S = (M - \tfrac{1}{2}J)^{-1}(M + \tfrac{1}{2}J). \tag{22}$$

We have the properties

$$M(S^{-1}) = -M(S) \tag{23}$$

and, when in addition $S', SS' \in \mathrm{Sp}_{(0)}(2n,\mathbb{R})$:

$$M(SS') = M(S) + (S^T - I)^{-1}J(M(S) + M(S'))^{-1}J(S-I)^{-1}. \tag{24}$$

3.1.2 The intertwining operators $R_\tau(S)$

We will need the following well-known generalization of the Fresnel formula (see e.g. Folland [6], Appendix A): let X be a real invertible matrix of dimension m; then:

$$\int e^{-2\pi iuv} e^{i\pi Xv^2} dv = |\det X|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sign} X} e^{-i\pi X^{-1}u^2} \tag{25}$$

where $\operatorname{sign} X$ is the difference between the number of > 0 and < 0 eigenvalues of X. Using this formula and the two lemmas above we set out to study the operators

$$R_\tau(S) = \sqrt{|\det(S-I)|} \int \widehat{T}_\tau(Sz)\widehat{T}_\tau(-z)dz \tag{26}$$

defined for $S \in \mathrm{Sp}_{(0)}(2n,\mathbb{R})$.

Proposition 4 *(i) Let $S \in \mathrm{Sp}_{(0)}(2n, \mathbb{R})$. The operator $R_\tau(S)$ is a continuous mapping $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ satisfying*

$$R_\tau(S)\widehat{T}_\tau(z) = \widehat{T}_\tau(Sz)R_\tau(S) \tag{27}$$

and we have

$$R_\tau(S)\operatorname{Op}_\tau(a) = \operatorname{Op}_\tau(a \circ S)R_\tau(S). \tag{28}$$

(ii) Let S, S', $SS' \in \mathrm{Sp}_{(0)}(2n, \mathbb{R})$. We have

$$R_\tau(SS') = e^{i\frac{\pi}{4}\operatorname{sign} M(SS')}R_\tau(S)R_\tau(S') \tag{29}$$

(iii) The operator (26) satisfies

$$R_\tau(S^{-1}) = R_\tau(S)^{-1} = R_{1-\tau}(S)^* \tag{30}$$

Proof. (i) It is equivalent to show that the operators

$$\Gamma_\tau(S) = \int \widehat{T}_\tau(Sz)\widehat{T}_\tau(-z)dz$$

are such that $\Gamma_\tau(S)\widehat{T}_\tau(z) = \widehat{T}_\tau(Sz)\Gamma_\tau(S)$. Let $f \in \mathcal{S}(\mathbb{R}^n)$; in view of formula (13) we have

$$\Gamma_\tau(S)f = \int e^{i\pi\sigma(Sz,z)}\widehat{T}_\tau((S-I)z)f dz;$$

since $S - I$ is a linear automorphism, $\widehat{T}_\tau((S-I)z) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ hence $\Gamma_\tau(S)f \in \mathcal{S}(\mathbb{R}^n)$. The continuity of $\Gamma_\tau(S)$ is straightforward to verify. Set

$$F(z, z_0) = \widehat{T}_\tau(Sz)\widehat{T}_\tau(-z)\widehat{T}_\tau(z_0)$$
$$G(z, z_0) = \widehat{T}_\tau(Sz_0)\widehat{T}_\tau(Sz)\widehat{T}_\tau(-z).$$

By repeated use of formula (13) one gets

$$F(z, z_0) = e^{-i\pi\sigma(Sz - z_0, z - z_0)}\widehat{T}_\tau((S-I)z + z_0)$$
$$G(z, z_0) = e^{-i\pi\sigma((S-I)z_0 + Sz_0, z)}\widehat{T}_\tau((S-I)z + Sz_0)$$

hence $G(z - z_0, z_0) = F(z, z_0)$. It follows that $\int F(z, z_0)dz = \int G(z, z_0)dz$ hence the equality (27). That the operators $R_\tau(S)$ satisfy the intertwining relation (28) follows using definition (14) of $\operatorname{Op}_\tau(a)$. (ii) (Cf. the proof of Proposition 4.2 in de Gosson [10]). For brevity we write $M = M(S)$,

$M' = M(S')$. In view of the composition formula (17) the twisted symbol c_σ of $R_\tau(S)R_\tau(S')$ is given by

$$c_\sigma(z) = K \int e^{i\pi[\sigma(z,z')+\Phi(z,z')]} dz'$$

where the constant K and the phase Φ are given by

$$\begin{aligned} K &= |\det(S-I)(S'-I)|^{-1/2} \\ \Phi(z,z') &= Mz^2 - 2Mz\cdot z' + (M+M')z'^2 \end{aligned}$$

A straightforward calculation shows that

$$\sigma(z,z') - 2Mz\cdot z' = -2J(S-I)^{-1}z\cdot z'$$

hence

$$\sigma(z,z') + \Phi(z,z') = -2J(S-I)^{-1}z\cdot z' + Mz^2 + (M+M')z'^2.$$

It follows that

$$c_\sigma(z) = Ke^{i\pi Mz^2} \int e^{-2\pi iJ(S-I)^{-1}z\cdot z'} e^{i\pi(M+M')z'^2} dz'. \tag{31}$$

Applying the Fresnel formula (25) with $X = M + M'$ to the formula above and replacing K with its value we get

$$c_\sigma(z) = |\det[(M+M')(S-I)(S'-I)]|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sign}(M+M')} e^{2\pi i\Theta(z)} \tag{32}$$

where the phase Θ is given by

$$\begin{aligned} \Theta(z) &= \left[M + (S^T - I)^{-1}J(M+M')^{-1}J(S-I)^{-1}\right] z^2 \\ &= M(SS')z^2 \end{aligned}$$

(the second equality in view of formula (24)). Noting that by definition of the symplectic Cayley transform we have

$$M + M' = J(I + (S-I)^{-1} + (S'-I)^{-1})$$

it follows, using property (24) of the symplectic Cayley transform, that

$$\begin{aligned} \det[(M+M')(S-I)(S'-I)] &= \det[(S-I)(M+M')(S'-I)] \\ &= \det[(S-I)(M+M')(S'-I)] \\ &= \det(SS'-I) \end{aligned}$$

which concludes the proof of the first part of proposition. (iii) Let us first show that $R_\tau(S^{-1}) = R_\tau(S)^{-1}$. Let c be the symbol of $C = R_\tau(S)R_\tau(S^{-1})$; we claim that $c_\sigma(z) = \delta(z)$, hence $C = I$. Noting that $\det(S^{-1} - I) = \det(S - I) \neq 0$, formula (31) in the proof of part (ii) shows that

$$c_\sigma(z) = Le^{i\pi M z^2} \int e^{-2\pi i J(S-I)^{-1} z\cdot z'} e^{i\pi(M+M(S^{-1}))z'^2} dz'$$

where $L = |\det(S - I)|^{-1}$. Since $M(S^{-1}) = -M$ we have, setting $z'' = (S^T - I)^{-1} J z'$,

$$\begin{aligned}
&= \frac{e^{i\pi M z^2}}{|\det(S - I)|} \int e^{-2\pi i J(S-I)^{-1} z\cdot z'} dz' \\
&= e^{i\pi M z^2} \int e^{2\pi i z z''} dz''
\end{aligned}$$

hence $c_\sigma(z) = \delta(z)$ by the Fourier inversion formula, which proves our claim. Let us finally show that $R_\tau(S^{-1}) = R_{1-\tau}(S)^*$. We have

$$\begin{aligned}
R_\tau(S^{-1}) &= \frac{1}{\sqrt{|\det(S^{-1} - I)|}} \int e^{i\pi M(S^{-1}) z^2} \widehat{T}_\tau(z) dz \\
&= \frac{1}{\sqrt{|\det(S - I)|}} \int e^{-i\pi M(S) z^2} \widehat{T}_\tau(z) dz
\end{aligned}$$

hence, using formula (7) for the adjoint of a τ-operator,

$$R_\tau(S^{-1})^* = \frac{1}{\sqrt{|\det(S - I)|}} \int e^{i\pi M(S) z^2} \widehat{T}_{1-\tau}(z) dz = R_{1-\tau}(S)$$

which is the same thing as $R_\tau(S^{-1}) = R_{1-\tau}(S)^*$. ∎

Notice that formula (30) shows that the operators $R_\tau(S)$ are unitary if and only if $\tau = \frac{1}{2}$ (the Weyl case) see de Gosson [8, 9, 10, 11].

3.1.3 Application to the τ-Wigner function

The usual cross-Wigner function $W(f, g)$ has the following well-known (and very useful) property of symplectic covariance: for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $S \in \mathrm{Sp}(2n, \mathbb{R})$ we have

$$W(\widehat{S}f, \widehat{S}g)(z) = W(f, g)(S^{-1}z) \tag{33}$$

where $\widehat{S} \in \mathrm{Mp}(2n, \mathbb{R})$ is any of the two metaplectic operators which cover S. In the τ-dependent case this result must be modified as follows:

Proposition 5 *Let $S \in \mathrm{Sp}_{(0)}(2n,\mathbb{R})$ and $f,g \in \mathcal{S}(\mathbb{R}^n)$. We have*

$$W_\tau(R_\tau(S)f, R_{1-\tau}(S)g)(z) = W_\tau(f,g)(S^{-1}z). \tag{34}$$

Proof. Let $A_\tau = \mathrm{Op}_\tau(a)$. Recall that $(\mathrm{Op}_\tau(a)f|g)_{L^2} = \langle a|W_\tau(f,g)\rangle$ (formula (18). In view of the second equality (30) we have, using

$$\begin{aligned}(R_\tau(S)\,\mathrm{Op}_\tau(a)f|R_{1-\tau}(S)g)_{L^2} &= (R_{1-\tau}(S)^* R_\tau(S)\,\underset{\tau}{\mathrm{Op}}(a)f|g)_{L^2} \\ &= (\mathrm{Op}_\tau(a)f|g)_{L^2} \\ &= \langle a|W_\tau(f,g)\rangle.\end{aligned}$$

On the other hand, using the intertwining property (28), we have

$$\begin{aligned}(R_\tau(S)\,\mathrm{Op}_\tau(a)f|R_{1-\tau}(S)g)_{L^2} &= (\mathrm{Op}_\tau(a\circ S)R_\tau(S)f|R_{1-\tau}(S)g)_{L^2} \\ &= \langle a\circ S|W_\tau(R_\tau(S)f, R_{1-\tau}(S)g)\rangle \\ &= \langle a, W_\tau(R_\tau(S)f, R_{1-\tau}(S)g)\circ S^{-.1}\rangle\end{aligned}$$

(the last identity using the change of variables $z \longmapsto S^{-1}z$ and the fact that $\det S = 1$). Formula (34) follows. ■

3.2 The operators $R_\tau(S)$ as pseudo-differential operators

Following result identifies $R_\tau(S)$ as a τ-pseudo-differential operator:

Proposition 6 *Let $S \in \mathrm{Sp}_{(0)}(2n,\mathbb{R})$; we have*

$$R_\tau(S) = \int s_\sigma(z)\widehat{T}_\tau(z)dz \tag{35}$$

where $s_\sigma(z)$ is given by the formula

$$s_\sigma(z) = \frac{1}{\sqrt{|\det(S-I)|}}e^{i\pi M(S)z^2}; \tag{36}$$

Proof. We have (see the proof of Proposition 4)

$$R_\tau(S) = \int e^{i\pi\sigma(Sz,z)}\widehat{T}_\tau((S-I)z)dz$$

the change of variables $z' = (S-I)z$ yields

$$\Gamma_\tau(S) = |\det(S-I)|^{-1}\int e^{i\pi\sigma(S(S-I)^{-1}z',z')}\widehat{T}_\tau(z')dz'.$$

Since $S(S-I)^{-1} = I + (S-I)^{-1}$ we have

$$\begin{aligned}
\sigma(S(S-I)^{-1}z', z') &= \sigma((S-I)^{-1}z', z') \\
&= J(S-I)^{-1}z' \cdot z' \\
&= \tfrac{1}{2}J + J(S-I)^{-1} \\
&= M(S)
\end{aligned}$$

hence (36). ∎

For practical calculations formula (15) is useful; it immediately yields: The distributional kernel of the operator $R_\tau(S)$ satisfies

$$K_\tau(S)(x+y, y) = \frac{1}{\sqrt{|\det(S-I)|}} \int e^{2\pi i(\tau x + y)p} e^{i\pi M(S)z^2} dp; \tag{37}$$

this formula can be used in principle for the calculation of explicit expressions for the operators $R_\tau(S)$. Let us give an example. Choosing $S = J$ we have $M(S) = \frac{1}{2}I$; a straightforward computation using (37) yields

$$K_\tau(J)(x, y) = e^{i\frac{n\pi}{4}} e^{\frac{i\pi}{2}(x-y)^2} e^{-2\pi(\tau x + (1-\tau)y)^2}. \tag{38}$$

Notice that when $\tau = \frac{1}{2}$ we get

$$K_{1/2}(J)(x, y) = e^{i\frac{n\pi}{4}} e^{-2\pi xy} \tag{39}$$

hence $R_{1/2}(J)$ is, up to a factor, the usual Fourier transform. In fact we have $R_{1/2}(J) \in \mathrm{Mp}(2n, \mathbb{R})$; it is the metaplectic operator $\widehat{J}$ with projection J on the symplectic group (see e.g. de Gosson [8], Folland [6]). This is not pure coincidence, in fact:

Corollary 7 *For $S \in \mathrm{Sp}_{(0)}(2n, \mathbb{R})$ the operators $R(S) = R_{1/2}(S)$ are, up to a unimodular factor $i^{\nu(S)}$ elements of the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$. In fact, when $\nu(S)$ is, modulo 2, the Conley–Zehnder of a any path joining the identity to S then $i^{\nu(S)}R(S) \in \mathrm{Mp}(2n, \mathbb{R})$.*

Proof. In [8, 9, 10, 11] we have shown that

$$R^\nu(S) = \frac{i^{\nu(S)}}{\sqrt{|\det(S-I)|}} \int e^{i\pi M(S)z^2} \widehat{T}(z) dz \tag{40}$$

when $\nu(S)$ is the Conley–Zehnder [5] index (which we discuss below). The result follows since $\widehat{T}_{1/2}(z) = \widehat{T}(z)$. ∎

4 The Case of Born–Jordan Operators

4.1 Born–Jordan operators

4.1.1 Motivation

Concurrently with Weyl, the physicists Born and Jordan [4] elaborated on Heisenberg's seminal paper [13] on "matrix mechanics" and proposed the quantization rule

$$x_j^m p_j^\ell \overset{\mathrm{BJ}}{\longrightarrow} \frac{1}{\ell+1}\sum_{k=0}^{\ell} \widehat{P_j}^{\ell-k}\widehat{X_j}^m\widehat{P_j}^k \tag{41}$$

which coincides with (1) when $m+\ell \leq 2$. We now make the following fundamental remark: the Born–Jordan prescription (41) is obtained by averaging the τ-ordering (2) on the interval $[0,1]$; this is immediately seen using the property

$$B(k+1,\ell-k+1) = \int_0^1 (1-\tau)^k \tau^{\ell-k} d\tau = \frac{k!(\ell-k)!}{(k+\ell+1)!}$$

of the beta function. This suggests to study, more generally, the pseudo-differential operators

$$A_{\mathrm{BJ}} = \int_0^1 A_\tau d\tau.$$

4.1.2 Definition of Born–Jordan operators

In [1] Boggiatto et al. define a transform $Q : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ by integrating over $[0,1]$ the τ-cross Wigner transforms (19); we will use the notation $Q = W_{\mathrm{BJ}}$; thus, for $f,g \in \mathcal{S}(\mathbb{R}^n)$:

$$W_{\mathrm{BJ}}(f,g) = \int_0^1 W_\tau(f,g)d\tau. \tag{42}$$

For $a \in \mathcal{S}'(\mathbb{R}^n)$ these authors define an operator, which we denote A_{BJ}, by the formula

$$(A_{\mathrm{BJ}}f|g)_{L^2} = \langle a, W_{\mathrm{BJ}}(f,g)\rangle \tag{43}$$

$f,g \in \mathcal{S}(\mathbb{R}^n)$ (cf. (18)). We will call A_{BJ} the Born–Jordan operator with symbol a and write $A_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a)$.

Using the representation (14) of the τ-operators we have

$$A_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a) = \int a_\sigma(z)\widehat{T}_{\mathrm{BJ}}(z)dz \tag{44}$$

where $\widehat{T}_{\mathrm{BJ}}(z)$ is the unitary operator defined by

$$\widehat{T}_{\mathrm{BJ}}(z) = \int_0^1 \widehat{T}_\tau(z_0) d\tau. \tag{45}$$

We notice that it immediately from the relations (12)–(13) that:

$$\widehat{T}_{\mathrm{BJ}}(z_0)\widehat{T}_{\mathrm{BJ}}(z_1) = e^{2\pi i\sigma(z_0,z_1)}\widehat{T}_{\mathrm{BJ}}(z_1)\widehat{T}_{\mathrm{BJ}}(z_0) \tag{46}$$

$$\widehat{T}_{\mathrm{BJ}}(z_0+z_1) = e^{-i\pi\sigma(z_0,z_1)}\widehat{T}_{\mathrm{BJ}}(z_0)\widehat{T}_{\mathrm{BJ}}(z_1). \tag{47}$$

Proposition 8 *The Born–Jordan operator $A_{BJ} = \mathrm{Op}_{\mathrm{BJ}}(a)$ is given by*

$$A_{BJ} = \int a_\sigma(z)\Theta(z)\widehat{T}(z)dz \tag{48}$$

where Θ is the real function defined by

$$\Theta(z) = \frac{\sin(2\pi px)}{2\pi px}. \tag{49}$$

The operator A_{BJ} is a continuous operator $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ for every $a \in \mathcal{S}'(\mathbb{R}^{2n})$. That this formula really defines a continuous operator $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ follows from the fact that $\Theta \in L^\infty(\mathbb{R}^{2n})$.

Proof. Integrating both sides of formula (9) in the interval $[0,1]$ we have $\widehat{T}_{\mathrm{BJ}}(z) = \Theta(z)\widehat{T}(z)$ hence the expression (48). ■

In view of the relation (7) between a τ-operator and its adjoint we have

$$\mathrm{Op}_{\mathrm{BJ}}(a)^* = \mathrm{Op}_{\mathrm{BJ}}(\overline{a}) \tag{50}$$

hence the Born–Jordan operators share with Weyl operators the property that they are (formally) self-adjoint if and only if their symbol is real. This makes Born–Jordan operators good candidates for quantization.

The reader is urged to notice that while every Born–Jordan operator is a Weyl operator, the converse property is not true because an arbitrary distribution $b_\sigma \in \mathcal{S}'(\mathbb{R}^{2n})$ cannot in general be written in the form $a_\sigma\Theta$ (see de Gosson and Luef [12]; also the discussion in Kauffmann [14]).

4.2 Reduced metaplectic covariance

The intertwining properties for τ operators do not carry over to the Born–Jordan case; it is meaningless to expect a relation like $R_{\mathrm{BJ}}(S)\widehat{T}_{\mathrm{BJ}}(z) = \widehat{T}_{\mathrm{BJ}}(Sz)R_{\mathrm{BJ}}(S)$ which would lead to a symplectic covariance property of the

type (28). The good news is, however, that Born–Jordan operators enjoy a symplectic covariance property for operators belonging to a subgroup of the standard metaplectic group $\mathrm{Mp}(2n,\mathbb{R})$. Recall that $\mathrm{Mp}(2n,\mathbb{R})$ is generated by the modified Fourier transform $\widehat{J}=e^{i\frac{n\pi}{4}}F$, the multiplication operators $\widehat{V}_{-P}f=e^{i\pi Px^2}f$ $(P=P^T)$ and the unitary scaling operators $\widehat{M}_{L,m}f(x)=i^m\sqrt{|\det L|}f(Lx)$ $(\det L\neq 0,\ m\pi=\arg\det L)$. The projections of these operators on $\mathrm{Sp}(2n,\mathbb{R})$ are, respectively, J, $V_{-P}=\begin{pmatrix} I & 0 \\ P & I\end{pmatrix}$, and $M_L=\begin{pmatrix} L^{-1} & 0 \\ 0 & L^2\end{pmatrix}$.

Proposition 9 *Let $A_{BJ}=\mathrm{Op}_{\mathrm{BJ}}(a)$ with $a\in\mathcal{S}'(\mathbb{R}^{2n})$. We have*

$$\widehat{S}\,\mathrm{Op}_{\mathrm{BJ}}(a)=\mathrm{Op}_{\mathrm{BJ}}(a\circ S^{-1})\widehat{S} \tag{51}$$

for every $\widehat{S}\in\mathrm{Mp}(2n,\mathbb{R})$ which is a product of a (finite number) of operators $\widehat{J}$ and $\widehat{M}_{L,m}$.

Proof. It suffices to prove formula (51) for $\widehat{S}=\widehat{J}$ and $\widehat{S}=\widehat{M}_{L,m}$. Let $\widehat{S}$ be anyone of these operators; we have

$$\begin{aligned}\widehat{S}\,\mathrm{Op}_{\mathrm{BJ}}(a)&=\int a_\sigma(z)\Theta(z)\widehat{S}\widehat{T}(z)dz\\ &=\left(\int a_\sigma(z)\Theta(z)\widehat{T}(Sz)dz\right)\widehat{S}\end{aligned}$$

where the second equality follows from the usual symplectic covariance property $\widehat{S}\widehat{T}(z)=\widehat{T}(Sz)\widehat{S}$ of the Heisenberg operators. Making the change of variables $z'=Sz$ in the integral we get, since $\det S=1$,

$$\int a_\sigma(z)\Theta(z)\widehat{T}(Sz)dz=\int a_\sigma(S^{-1}z)\Theta(S^{-1}z)\widehat{T}(z)dz.$$

Now, by definition of the symplectic Fourier transform we have

$$a_\sigma(S^{-1}z)=\int e^{-2\pi i\sigma(S^{-1}z,z')}a(z')dz'=(a\circ S^{-1})_\sigma(z).$$

On the other hand

$$\Theta(M_L^{-1}z)=\frac{\sin(2\pi Lp\cdot(L^T)^{-1}x)}{2\pi Lp\cdot(L^T)^{-1}x}=\Theta(z)$$

and, similarly, $\Theta(J^{-1}z) = \Theta(z)$ so we have

$$\begin{aligned}\widehat{S}\,\mathrm{Op}_{\mathrm{BJ}}(a) &= \left(\int (a \circ S^{-1})_{\sigma}\Theta(z)\widehat{T}(z)dz\right)\widehat{S} \\ &= \mathrm{Op}_{\mathrm{BJ}}(a \circ S^{-1})\widehat{S}\end{aligned}$$

whence formula (51). ■

The proof above shows that the essential step consists in noting that $\Theta(S^{-1}z) = \Theta(z)$ when $S = J$ or $S = M_L$. It is clear that this property fails if one takes $S = V_P$ with $P \neq 0$, so we cannot expect to have full symplectic covariance for Born–Jordan operators. Such a property is anyway excluded in view of our discussion in the Introduction to this paper. symplectic covariance is characteristic of Weyl calculus.

References

[1] Boggiatto P., De Donno G., Oliaro A., Time-Frequency Representations of Wigner Type and Pseudo-Differential Operators, Transactions of the Amer. Math. Soc., **362**(9) 4955–4981 (2010)

[2] Boggiatto P., Bui Kien Cuong, De Donno, G., Oliaro A., Weighted integrals of Wigner representations, Journal of Pseudo-Differential Operators and Applications, 2010

[3] Boggiatto P., De Donno G., Oliaro A., Hudson Theorem for τ-Wigner Transforms, Quaderni scientifici del Dipartimento di Matematica, Univeristà di Torino, Quaderno N. 1, 2010

[4] Born, M., Jordan, P., Zur Quantenmechanik, Z. Physik 34, 858–888 (1925)

[5] Conley C., Zehnder, E., Morse-type index theory for flows and periodic solutions of Hamiltonian equations. *Comm. Pure and Appl. Math.* **37** (1984) 207–253

[6] Folland, G.B., Harmonic Analysis in Phase space. Annals of Mathematics studies, Princeton University Press, Princeton, N.J. (1989)

[7] de Gosson M., The Weyl Representation of Metaplectic operators. *Letters in Mathematical Physics* **72** (2005) 129–142

[8] Symplectic Geometry and Quantum Mechanics, Birkhäuser, Basel, series "Operator Theory: Advances and Applications" (subseries: "Advances in Partial Differential Equations"), Vol. 166 (2006)

[9] de Gosson M., de Gosson, S., An extension of the Conley–Zehnder Index, a product formula and an application to the Weyl representation of metaplectic operators . *J. Math. Phys.,* **47**(12), 123506, 15 pages, (2006)

[10] de Gosson M., Metaplectic Representation, Conley–Zehnder Index, and Weyl Calculus on Phase Space. *Rev. Math. Physics*, **19**(8) (2007) 1149–1188

[11] de Gosson M., On the usefulness of an index due to Leray for studying the intersections of Lagrangian and symplectic paths. J. Math. Pures Appl. **91** (2009) 598–613

[12] de Gosson, M., Luef, F., Preferred Quantization Rules: Born–Jordan vs. Weyl; Applications to Phase Space Quantization. *J. Pseudo-Differ. Oper. Appl.*, 2(1) (2011) 115–139

[13] Heisenberg, W., Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen, Zeitschrift für Physik, 33, 879–893 (1925) [English translation in: B.L. van der Waerden, editor, Sources of Quantum Mechanics, Dover Publications (1968)

[14] Kauffmann, S.K. , Unambiguous quantization from the maximum classical correspondence that is self-consistent: the slightly stronger canonical commutation rule Dirac missed, Found. Phys. 41, 805–918 (2011)

[15] Leray, J., *Lagrangian Analysis and Quantum Mechanics*, *a mathematical structure related to asymptotic expansions and the Maslov index* (the MIT Press, Cambridge, Mass., 1981); translated from *Analyse Lagrangienne* RCP 25, Strasbourg Collège de France, 1976–1977

[16] Reiter, H., Metaplectic Groups and Segal Algebras, Springer (1989)

[17] Shubin, M.A., Pseudodifferential Operators and Spectral Theory, Springer–Verlag, (1987) [original Russian edition in Nauka, Moskva (1978)]

[18] Stein, E.M., *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press (1993)

[19] Weil, A., Sur certains groupes d'opérateurs unitaires, *Acta Math.* **111**, 143–211 (1964); also in *Collected Papers*, Vol. III:1–69, Springer-Verlag, Heidelberg, 1980.

[20] Weyl, H., Quantenmechanik und Gruppentheorie, Zeitschrift für Physik, 46 (1927)

[21] Wong, M.W., Weyl Transforms, Springer (1998)